An intrinsic formulation of nonrelativistic analytical mechanics and wave mechanics

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INTRODUCTION

As is well known Newton's equations do not depend on the inertial frame chosen. A geometric formulation of Newtonian mechanics in Newtonian space-time is therefore possible. On the other hand intrinsic formulation of analytical mechanics in Newtonian space-time is not possible since different Lagrangians and different Hamiltonians are used for different inertial frames. The same is true of the Hamilton-Jacobi theory and Schrödinger wave mechanics. A similar difficulty encountered in analytical mechanics of charged particles was resolved by enlarging the configuration manifold of the particle [6]. The same technique is used in the present note. An intrinsic formulation of analytical mechanics and wave mechanics is obtained. Conventional formulations are obtained by reducing the dimension in a frame dependent way. This work is related to the papers of Bergmann [1], Lévy-Leblond [5] and Duval and Künzle [4].

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1. AFFINE SPACES AND AFFINE TRANSFORMATIONS

An *affine space* is a triple (M, V, ρ), where M is a set, V is a vector space, ρ is

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a mapping of $M \times M$ in V and the following conditions are satisfied:

(a) $\rho(x, x) = 0$,

(b) $\rho(x'', x') + \rho(x', x) = \rho(x'', x)$,

(c) for each x the mapping $\rho_x : M \to V$ defined by $\rho_x(x') = \rho(x', x)$ is bijective.

We will write $x' - x$ instead of $\rho(x', x)$ and $x + v$ instead of $\rho_x^{-1}(v)$.

Automorphisms of an affine space are called *affine transformations*. More specifically, a bijective mapping $\alpha : M \to M$ is an affine transformation if there exists a linear transformation $a : V \to V$ such that $\alpha(x') - \alpha(x) = a(x' - x)$. The linear transformation a is called the *linear part* of the affine transformation α. An affine transformation is called a *translation* if its linear part is the identity mapping. The group of affine transformations of M will be denoted by $A(M)$ and the group of linear transformations of V will be denoted by $GL(V)$. For each point x in M there is a bijective relation between elements of $A(M)$ and $GL(V) \times \times V$. The affine transformation $\alpha = (a, c)_x$ corresponding to the pair (a, c) is defined by

$$\alpha(x') = x + a(x' - x) + c.$$

The linear transformation a is the linear part of the affine transformation $\alpha = = (a, c)_x$ and the vector c is obtained from

$$c = \alpha(x) - x.$$

The equality

$$(a', c')_{x'} = (a, c)_x$$

implies

$$a' = a, \quad c' = c + a(x' - x) - (x' - x).$$

The composition relation

$$(a', c')_x \circ (a, c)_x = (a' \circ a, a'(c) + c')_x$$

is easily verified.

2. THE NEWTONIAN SPACE-TIME AND GALILEI TRANSFORMATIONS

Let (M, V, ρ) be an affine space of dimension 4 representing the physical space--time of nonrelativistic physics. Let τ be an element of the space V^* dual to V. The set

$$T_0 = \{v \in V; \langle v, \tau \rangle = 0\}$$

is a vector subspace of V and

$$T_1 = \{ v \in V; \langle v, \tau \rangle = 1 \}$$

is an affine subspace of V. If $\theta : T_1 \times T_1 \to T_0$ is defined by $\theta(v', v) = v' - v$ then (T_1, T_0, θ) is an affine space. Let $g : T_0 \to T_0^*$ be an Euclidean metric tensor on T_0. The objects τ and g are used to measure time intervals and spacial distances. The time Δt elapsed between two events x and x' in M is expressed by

$$\Delta t = \langle x' - x, \tau \rangle.$$

The two events are simultaneous if $\langle x' - x, \tau \rangle = 0$. The distance d between simultaneous events x and x' is calculated from

$$d^2 = \langle x' - x, g(x' - x) \rangle.$$

The system (M, V, ρ, τ, g) is called the *Newtonian space-time*. An *inertial frame* in the Newtonian space-time is an element u of the affine subspace T_1.

Automorphisms of the Newtonian space-time are called *Galilei transformations*. The group of Galilei transformations will be denoted by $GA(M)$. A Galilei transformation is an affine transformation α of M whose linear part a is an automorphism of the system (V, τ, g). Restricted to T_0 the linear transformation a induces a linear transformation a_0 of T_0. This transformation belongs to the group $O(T_0)$ of automorphisms of (T_0, g) called orthogonal transformations. Restricted to T_1 the transformation a induces an affine transformation a_1 of T_1. The orthogonal transformation a_0 is the linear part of a_1. The linear transformation a is completely determined by the affine transformation a_1. It u is any inertial frame then

$$a(v) = a(v + (1 - \langle v, \tau \rangle)u) - (1 - \langle v, \tau \rangle) a(u) =$$

$$= a_1(v + (1 - \langle v, \tau \rangle)u) - (1 - \langle v, \tau \rangle) a_1(u).$$

It follows that if an inertial frame u is chosen then elements of the group $GA(V)$ of automorphisms of (V, τ, g) are in one to one correspondence with elements of $O(T_0) \times T_0$. We will denote by $(a_0, b)_u$ the automorphism corresponding to the pair (a_0, b). If $a = (a_0, b)_u$ then a_0 is the orthogonal transformation of T_0 induced by a and $b = a(u) - u$. Hence,

$$a(v) = a_0(v - \langle v, \tau \rangle u) + \langle v, \tau \rangle (u + b).$$

The equality

$$(a'_0, b')_{u'} = (a_0, b)_u$$

implies

$$a'_0 = a_0, \quad b' = b + a_0(u' - u) - u' + u.$$

If $a = (a_0, b)_u$ and $a' = (a'_0, b')_u$ then

$$a' \circ a = (a'_0 \circ a_0, a'_0(b) + b')_u .$$

A choice of an event in M introduces a one to one correspondence between Galilei transformations and elements of $GA(V) \times V$. If an inertial frame is chosen then elements of $GA(V)$ are in one to one correspondence with elements of $O(T_0) \times T_0$. If both an event x and an inertial frame u are chosen then we have a bijective relation between Galilei transformations and elements of $O(T_0) \times T_0 \times V$. We will denote by $(a_0, b, c)_{(x,u)}$ the Galilei transformation $((a_0, b)_u, c)_x$ corresponding to the triple (a_0, b, c). If $\alpha = (a_0, b, c)_{(x,u)}$ then

$$\alpha(x') = x + (a_0((x' - x) - \langle x' - x, \tau\rangle u) + \langle x' - x, \tau\rangle(u + b) + c).$$

The equality

$$(a'_0, b', c')_{(x',u')} = (a_0, b, c)_{(x,u)}$$

implies

$$a'_0 = a_0, \quad b' = b + a_0(u' - u) - u' + u$$

and

$$c' = c + a_0((x' - x) - \langle x' - x, \tau\rangle u) + \langle x' - x, \tau\rangle(u + b) - (x' - x).$$

If $\alpha' = (a'_0, b', c')_{(x,u)}$ and $\alpha = (a_0, b, c)_{(x,u)}$ then

$$\alpha' \circ \alpha = (a'_0 \circ a_0, a'_0(b) + b',$$
$$a'_0(c - \langle c, \tau\rangle u) + \langle c, \tau\rangle(u + b) + c')_{(x,u)} .$$

3. NEWTONIAN MECHANICS

Newton's equations of motion are second order differential equations in the Newtonian space-time. Since M is an affine space the tangent bundle TM is identified with $M \times V$ and the second tangent bundle T^2M is identified with $M \times V \times V$. The cotangent bundle T^*M is identified with $M \times V^*$. The differential of a differentiable function $U : M \to \mathbb{R}$ will be interpreted as a mapping $dU : M \to V^*$. Let $\iota : T_0 \to V$ denote the canonical injection. A degenerate contravariant metric tensor $g' : V^* \to V$ is defined by $g' = \iota \circ g^{-1} \circ \iota^*$. The dynamics of a particle of mass m is described by the set

$$E = \{(x, \dot{x}, \ddot{x}) \in T^2M; \langle \dot{x}, \tau\rangle = 1, m\ddot{x} = -g'(dU(x))\},$$

where U is the potential energy of the particle.

4. INERTIAL FRAMES AND ANALYTICAL MECHANICS

Analytical mechanics of a particle is described by first order differential equations in the space-time-momentum-energy space T^*M identified with $M \times V^*$. An element p of V^* represents momentum and energy of a particle. If u is an inertial frame then $p - \langle u, p\rangle \tau$ is the momentum and $e = -\langle u, p\rangle$ is the energy. The tangent bundle TT^*M is identified with $M \times V^* \times V \times V^*$. In an inertial frame u the dynamics of a particle is represented by a submanifold D of TT^*M. An element (x, p, x', p') of TT^*M belongs to D if

$$x' = \langle x', \tau\rangle \left(u + \frac{1}{m} g'(p)\right),$$

$$p' = -\langle x', \tau\rangle \, dU(x)$$

and

$$\langle u, p\rangle + \frac{1}{2m} \langle g'(p), p\rangle + U(x) = 0.$$

The submanifold D is a homogenous Hamiltonian system [3] generated by a submanifold K of T^*M. An element (x, p) of T^*M belongs to K if

$$\langle u, p\rangle + \frac{1}{2m} \langle g'(p), p\rangle + U(x) = 0.$$

The tangent bundle TM is identified with $M \times V$. The homogenous Lagrangian for the system D is the function $L : TM \to \mathbb{R}$ defined by

$$L(x, x') = \frac{m}{2\langle x', \tau\rangle} \langle x' - \langle x', \tau\rangle u, g(x' - \langle x', \tau\rangle u)\rangle - \langle x', \tau\rangle U(x).$$

It is easily seen that submanifolds D corresponding to different inertial frames u and u' are different. It is not possible to formulate analytical mechanics in Newtonian space-time in a frame independent way. A frame independent formulation of analytical mechanics is possible in an enlarged space called the Galilei space.

5. HAMILTON-JACOBI EQUATIONS IN THE NEWTONIAN SPACE-TIME

The Hamilton-Jacobi equation in an inertial frame u is the equation

$$\langle u, dS\rangle + \frac{1}{2m}\langle g' \circ dS, dS\rangle + U = 0$$

for a function $S : M \to \mathbb{R}$.

The Hamilton-Jacobi theory in the Newtonian space-time is frame dependent. An intrinsic Hamilton-Jacobi theory can be formulated in the Galilei space.

6. INERTIAL FRAMES AND SCHRÖDINGER EQUATIONS

Let u be an inertial frame. The wave function $\psi : M \to \mathbb{C}$ of a particle of mass m satisfies the Schrödinger equation

$$i\hbar\langle u, d\psi\rangle + \frac{\hbar^2}{2m}\Delta_{g'}\psi - U\psi = 0.$$

It is easily seen that a solution ψ of the Schrödinger equation in one inertial frame u will not in general satisfy the Schrödinger equation in a different inertial frame u'. The same quantum state of a particle must be represented by different wave functions in reference to different inertial frames. An intrinsic formulation of Schrödinger wave mechanics in Newtonian space-time is not possible. A frame independent formulation of wave mechanics in the Galilei space is described in Section 10. Conventional Schrödinger equations are obtained by frame dependent reductions.

7. THE GALILEI SPACE AND GALILEI TRANSFORMATIONS

The *Galilei space* is a system $(N, W, \sigma, \bar{g}, z)$, where (N, W, σ) is an affine space of dimension 5, $\bar{g} : W \to W^*$ is a metric tensor of signature 3 and z is a vector in W such that $\langle z, \bar{g}(z)\rangle = 0$.

Let $Z \subset W$ be the space of vectors proportional to z, let V denote the quotient space of W by Z and let $\chi : W \to V$ be the canonical projection. There is a unique element τ of V^* such that

$$\langle \chi(w), \tau\rangle = \langle w, \bar{g}(z)\rangle.$$

In the space $T_0 = \{v \in V; \langle v, \tau\rangle = 0\}$ there is a unique metric tensor $g : T_0 \to T_0^*$ such that

$$\langle \chi(w'), g(\chi(w))\rangle = \langle w', \bar{g}(w)\rangle$$

if $\langle w, \bar{g}(z)\rangle = 0$ and $\langle w', \bar{g}(z)\rangle = 0$. Let M denote the quotient space of N by the equivalence relation according to which two points y and y' are equivalent if $\sigma(y', y)$ belongs to Z. Let $\mu : N \to M$ be the canonical projection. There is a unique mapping $\rho : M \times M \to V$ such that

$$\rho(\mu(y'), \mu(y)) = \chi(\sigma(y', y)).$$

The system (M, V, ρ, τ, g) is the Newtonian space-time. We will write $y' - y$, $y + w$, $x' - x$ and $x + v$ instead of $\sigma(y', y)$, $\sigma_y^{-1}(w)$, $\rho(x', x)$ and $\rho_x^{-1}(v)$ respectively.

An *inertial frame* in the Galilei space is a vector $\overline{u}$ in W such that $\langle \overline{u}, \overline{g}(\overline{u})\rangle = 0$ and $\langle \overline{u}, \overline{g}(z)\rangle = 1$. The vector $u = \chi(\overline{u})$ is an inertial frame in the Newtonian space-time.

Automorphisms of the Galilei space will be called *extended Galilei transformations*. An extended Galilei transformation is an affine transformation $\overline{\alpha} : N \to N$ whose linear part $\overline{a} : W \to W$ is an automorphism of $(W, \overline{g}, z)$. The group of extended Galilei transformations will be denoted by $GA(N)$ and the group of automorphisms of $(W, \overline{g}, z)$ will be denoted by $GA(W)$. If $\overline{a}$ is an element of $GA(W)$ then there exists an element a of $GA(V)$ such that $\chi \circ \overline{a} = a \circ \chi$. Let $\overline{u}$ be an inertial frame. The space

$$U = \{w \in W; \langle w, \overline{g}(\overline{u})\rangle = 0\}$$

is a complement of Z in W. It follows that there exists a unique mapping $\kappa_u : V \to W$ such that $\chi(\kappa_u(v)) = v$ and $\langle \kappa_u(v), \overline{g}(\overline{u})\rangle = 0$. If $a = (a_0, b)_u$ then

$$\begin{aligned}\overline{a}(w) = {} & \kappa_u(a_0(\chi(w - \langle w, \overline{g}(z)\rangle \overline{u}))) + \\ & + \langle w, \overline{g}(z)\rangle \kappa_u(b) + \langle w, \overline{g}(z)\rangle \overline{u} + \\ & - \langle a_0(\chi(w - \langle w, \overline{g}(z)\rangle \overline{u})), g(b)\rangle z + \\ & - \frac{1}{2} \langle b, g(b)\rangle \langle w, \overline{g}(z)\rangle z + \langle w, \overline{g}(\overline{u})\rangle z.\end{aligned}$$

Since $\overline{a}$ is completely determined by a it follows that $GA(W)$ and $GA(V)$ are two canonically isomorphic faithful representations of the same abstract group. The choice of a point y in N establishes a one to one relation between extended Galilei transformations and the elements of $GA(W) \times W$. If $\overline{\alpha} = (\overline{a}, \overline{c})_y$ then

$$\overline{\alpha}(y') = y + \overline{a}(y' - y) + \overline{c}.$$

8. ANALYTICAL MECHANICS IN THE GALILEI SPACE

Spaces TN, T^*N and TT^*N are identified with $N \times W$, $N \times W^*$ and $N \times W^* \times W \times W^*$ respectively. We will denote by $\overline{g}' : W^* \to W$ the inverse of $\overline{g}$. The dynamics of a particle is described by a submanifold $\bar{D}$ of TT^*N. An element (y, r, y', r') belongs to $\bar{D}$ if

$$y' = \frac{1}{m} \langle y', \bar{g}(z) \rangle \bar{g}'(r) + \left\{ \frac{\langle y', \bar{g}(y') \rangle}{2\langle y', \bar{g}(z) \rangle} + \frac{1}{m} \langle y', \bar{g}(z) \rangle \bar{U}(y) \right\} z$$

and

$$r' = -\langle y', \bar{g}(z) \rangle \, d\bar{U}(y),$$

where $\bar{U} = U \circ \mu$. The submanifold $\bar{D}$ is a homogenous Hamiltonian system generated by a submanifold $\bar{K}$ of T^*N. An element (y, r) of T^*N belongs to $\bar{K}$ if

$$\frac{1}{2m} \langle \bar{g}'(r), r \rangle + \bar{U}(y) = 0$$

and

$$\langle z, r \rangle = m.$$

The homogenous Lagrangian for this system is the function $\bar{L} : TN \to \mathbb{R}$ defined by

$$\bar{L}(y, y') = \frac{m}{2\langle y', \bar{g}(z) \rangle} \langle y', \bar{g}(y') \rangle - \langle y', \bar{g}(z) \rangle \, \bar{U}(y).$$

9. THE HAMILTON-JACOBI THEORY IN THE GALILEI SPACE

The Hamilton-Jacobi theory in the Galilei space is based on the following two equations:

$$\frac{1}{2m} \langle \bar{g}' \circ dF, dF \rangle + \bar{U} = 0$$

and

$$\langle z, dF \rangle = m$$

for a function $F : N \to \mathbb{R}$.

10. WAVE MECHANICS IN THE GALILEI SPACE

Quantum states of a particle of mass m in the Galilei space are represented by wave functions $\phi : N \to \mathbb{C}$ satisfying wave equations

$$\frac{\hbar^2}{2m} \Delta_{\bar{g}'} \phi - \bar{U}\phi = 0$$

and

$$i\hbar\langle z, \mathrm{d}\phi\rangle + m\phi = 0,$$

where $\bar{U} = U \circ \mu$.

11. INERTIAL FRAMES AND REDUCTIONS IN ANALYTICAL MECHANICS

Constructions used in the present section are based on the theory of reductions of symplectic manifolds [2] adapted to the affine case.

Let C_0 and C_m be subspaces of W^* defined by

$$C_0 = \{ r \in W^*; \langle r, z\rangle = 0\}$$

and

$$C_m = \{ r \in W^*; \langle r, z\rangle = m\}.$$

The space C_0 is the polar of Z. It is also the image of χ^*. If a mapping $\omega : C_m \times C_m \to C_0$ is defined by $\omega(r', r) = r' - r$ then (C_m, C_0, ω) is an affine space. The space $N \times C_m$ is a coisotropic subspace of the symplectic affine space $T^*N = N \times W^*$. The reduced symplectic space is the affine space $M \times C_m$. The mapping $\mu \times \mathrm{id} : N \times C_m \to M \times C_m$ will be denoted by ν. The space $T(N \times C_m) = N \times C_m \times W \times C_0$ is a coisotropic subspace of the affine symplectic space $TT^*N = N \times W^* \times W \times W^*$. The reduced symplectic space is the space $T(M \times C_m) = M \times C_m \times V \times C_0$. The mapping $\mu \times \mathrm{id} \times \chi \times \mathrm{id} : N \times C_m \times W \times C_0 \to M \times C_m \times V \times C_0$ will be denoted by $T\nu$.

Reducing the Hamiltonion system $\bar{D}$ we obtain the submanifold

$$D^r = T\nu(\bar{D} \cap (M \times C_m \times V \times C_0)).$$

An element (x, r, x', r') of $M \times C_m \times V \times C_0$ is in D^r if

$$r' = -\langle x', \tau\rangle \chi^*(dU(x)),$$

$$x' = \frac{1}{2} \langle x', \tau\rangle \chi(\bar{g}'(r))$$

and

$$\frac{1}{2m} \langle \bar{g}'(r), r\rangle + U(x) = 0.$$

The submanifold D^r is a homogenous Hamiltonian system generated by the submanifold

$$K^r = \nu(\bar{K} \cap (M \times C_m)).$$

An element (x, r) of $M \times C_m$ is in K^r if

$$\frac{1}{2m} \langle \bar{g}'(r), r \rangle + U(x) = 0.$$

The Hamiltonian system D^r is not generated by a Lagrangian.

Let $\bar{u}$ be an inertial frame. The mapping $\kappa_u : W^* \to V^*$ restricted to C_m is bijective. Mappings

$$\lambda_u = \mathrm{id} \times \kappa_u^* \mid C_m : M \times C_m \to T^*M$$

and

$$T\lambda_u = \mathrm{id} \times \kappa_u^* \mid C_m \times \mathrm{id} \times \kappa_u^* \mid C_0 : M \times C_m \times V \times C_0 \to TT^*M$$

are symplectomorphisms. The submanifold $D = T\lambda_u(D^r)$ is the homogenous Hamiltonian system described in Section 4. It is generated by the submanifold $K = \lambda_u(K^r)$ and also by the homogenous Lagrangian L defined in Section 4.

Each of the objects $\bar{D}$, $\bar{K}$, $\bar{L}$, D^r and K^r provides an intrinsic representation of dynamics. Dynamics is also represented by equivalence classes of pairs (D, u), (K, u) or (L, u). If two pairs (D', u') and (D, u) are equivalent then

$$D' = (\mathrm{id} \times \eta \times \mathrm{id} \times \mathrm{id})(D),$$

where the mapping $\eta : V^* \to V^*$ is characterized by

$$\langle v, \eta(p) \rangle = \langle v, p \rangle - m \langle v - \langle v, \tau \rangle u, g(u' - u) \rangle + \\ + \frac{m}{2} \langle v, \tau \rangle \langle u' - u, g(u' - u) \rangle.$$

If pairs (K', u') and (K, u) are equivalent then

$$K' = (\mathrm{id} \times \eta)(K),$$

and if pairs (L', u') and (L, u) are equivalent then

$$L' = L \circ (\mathrm{id} \times \xi),$$

where the mapping $\xi : V \to V$ is defined by

$$\xi(x') = x' - \langle x', \tau \rangle (u' - u).$$

12. FRAME DEPENDENT REDUCTIONS IN THE HAMILTON-JACOBI THEORY

Let F be a function satisfying the Hamilton-Jacobi equations in the Galilei space. Let $\overline{u}$ be an inertial frame and let y be a point in N. The function $\overline{F}: N \rightarrow \mathbb{R}$ defined by

$$\overline{F}(y') = F(y') - m\langle y' - y, \overline{g}(\overline{u})\rangle$$

satisfies the condition

$$\langle z, d\overline{F}\rangle = 0.$$

It follows that there is a unique function $S: M \rightarrow \mathbb{R}$ such that

$$\overline{F} = S \circ \mu.$$

The function S satisfies the Hamilton-Jacobi equation

$$\langle u, dS\rangle + \frac{1}{2m}\langle g' \circ dS, dS\rangle + U = 0.$$

Each solution F of the intrinsic Hamilton-Jacobi equations can be represented by an equivalence class of triples (S, u, y), where u is an inertial frame, y is a point in N and S is a solution of the Hamilton-Jacobi equation corresponding to u. Two triples (S', u', y') and (S, u, y) are equivalent if

$$S' = S - \langle (x'' - x) - c - \langle (x'' - x) - c, \tau\rangle u, g(b)\rangle +$$
$$+ \frac{1}{2}\langle (x'' - x) - c, \tau\rangle\langle b, g(b)\rangle + \langle \overline{c}, \overline{g}(\overline{u})\rangle,$$

where $x = \mu(y)$, $\overline{c} = y' - y$, $c = \chi(\overline{c})$ and $b = \chi(\overline{u}' - u)$.

13. REDUCTIONS IN WAVE MECHANICS

Let ϕ be a wave function satisfying the wave equations in the Galilei space. Let $\overline{u}$ be an inertial frame and let y be a point in N. The function $\overline{\phi}$ defined on N by

$$\overline{\phi}(y') = \exp\left(-i\frac{m}{\hbar}\langle y' - y, \overline{g}(\overline{u})\rangle\right)\phi(y')$$

satisfies the equation

$$\langle z, \mathrm{d}\overline{\phi}\rangle = 0.$$

Consequently there is a unique function ψ on M such that

$$\overline{\phi} = \psi \circ \mu.$$

It easily shown that ψ satisfies the Schrödinger equation

$$i\hbar\langle u, \mathrm{d}\psi\rangle + \frac{\hbar^2}{2m}\Delta_{g'}\psi - U\psi = 0.$$

A quantum state is represented either by the wave function ϕ or by an equivalence class of triples (ψ, u, y), where u is an inertial frame, y is a point in N and ψ is a solution of the Schrödinger equation corresponding to u. Equivalent triples (ψ', u', y') and (ψ, u, y) are related by

$$\psi'(x'') = \exp\left(i\frac{m}{\hbar}\theta(x'' - x, b, \overline{c})\right)\psi(x''),$$

where

$$\theta(x'' - x, b, \overline{c}) = -\langle(x'' - x) - c - \langle(x'' - x) - c, \tau\rangle u, g(b)\rangle + \\ + \frac{1}{2}\langle(x'' - x) - c, \tau\rangle\langle b, g(b)\rangle + \langle\overline{c}, \overline{g}(\overline{u})\rangle,$$

and $x = \mu(y)$, $\overline{c} = y' - y$, $c = \chi(\overline{c})$, $b = \chi(\overline{u}' - u)$.

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REFERENCES

[1] V. BARGMANN, *On unitary ray representations of continuous groups*, Ann. Math., **59**, 1, (1954).

[2] S. BENENTI, *The Category of Symplectic Reductions*, Proceedings: International Meeting on Geometry and Physics, Pitagora Editrice Bologna, (1983).

[3] S. BENENTI and W.M. TULCZYJEW, *Sur le théorème de Jacobi en mécanique analytique*, C.R. Acad. Sc. Paris, **294**, 677 - 680, (1982).

[4] C. DUVAL and H.P. KUNZLE, *Minimal Gravitational Coupling in the Newtonian Theory and the Covariant Schrödinger Equation*, General Relativity and Gravitation, **16**, 333, (1984).

[5] J.-M. LÉVY-LEBLOND, *Nonrelativistic Particles and Wave Equations*, Commun. Math. Phys., **6**, 286, (1967).

[6] M.R. MENZIO and W.M. TULCZYJEW, *Infinitesimal symplectic relations and generalized Hamiltonian dynamics*, Ann. Inst. H. Poincaré, 28, 349, (1978).

[7] C. DUVAL, G. BURDET, H.P. KUNZLE and M. PERRIN, *Bargmann structures and Newton-Cartan theory*, Phys. Rev. D., **31**, 1841 (1985).

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